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Optimal Dual-Rate Digital Redesign with Application to Missile Control

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I. Introduction

DIGITAL flight-control systems can be obtained via digital redesign; that is, a known continuous-time (CT) autopilot is converted to discrete time (DT) for digital implementation. With digital redesign, the DT controllers are obtained either by discretizing the individual CT controllers[1] or by using a sophisticated method that takes into account the closed-loop topology and the dynamics of the system under control.[2] The availability of a CT control system prior to the selection of the sampling rate for digital control is a key feature of digital redesign. However, the selection of the sampling rates is constrained by the hardware selected for control, sensing, and actuation.[3] With single-rate DT systems, a high sampling rate is usually needed to guarantee closed-loop stability and performance, although, in practice, it may be inappropriate due to the risk of numerical errors and the unavailability of converters of sufficiently high resolution and computers of appropriate processing power, especially when design constraints on size, mass, and power consumption are present. Multirate sampling can be used to optimize the allocation of processing power and to allow greater flexibility in the design of multichannel, multiloop autopilots. Using plant-output sampling and control-input update at different rates can provide better trade offs between performance and implementation costs; for instance, a dual-rate control synthesis and implementation scheme makes it possible to effectively handle multiple dynamic scales and constraints on hardware effectively.[4]

This note proposes an optimal dual-rate digital-redesign method that can be applied to CT and high-rate DT control systems. Briefly, given the existence of a CT control system, or of a fast DT control system, a DT H_2 optimal-control problem is solved to convert the known control system to either a low-rate or a dual-rate digital control system in a way that guarantees closed-loop stability and performance in the DT H_2 sense. The idea of performing digital redesign using the H_2 method on the closed-loop system comes from Ref. 5, although the methods presented there yield single-rate digital control systems. In this Note, the proposed dual-rate digital-redesign technique results in digital control systems that have satisfactory closed-loop performance over an extended range of sampling rates, as compared with other widely used methods of digital redesign. Interestingly, the proposed dual-rate digital redesign applies to single- and multiloop systems. Furthermore, the proposed method of digital redesign makes use of generalized holds and samplers of the dual-rate type, thereby providing an added level of flexibility.[6] The proposed dual-rate digital-redesign technique is, however, constrained by the specific requirements that arise when a DT H_2 problem is solved.

II. Optimal Dual-Rate Digital Redesign

A. Assumptions

Assumption 1: The uniform sampling periods are h (low rate of $1/h$ Hz) and T (high rate of $1/T$ Hz). The periods are related as follows: $h = N \cdot T$, $N \in Z^+$, where Z^+ is the set of positive integers. T is chosen to be nonpathological[7] with respect to the plant transfer function.

Assumption 2: The hold device H and ideal sampler S, which can each take a period equal to T or h, are synchronized at time $t = 0$. The hold has a bounded response to a unit DT impulse input and does not introduce any discrete zero into the hold-equivalent model of the plant, which cancels a pole of the plant model at nonpathological T values.[8] For example, the zero-order hold (ZOH) satisfies this condition.

Assumption 3: Dual-rate digital redesign is performed with the objectives of preserving the step-input tracking performance[6] and the step-disturbance rejection property of the CT (or fast DT) closed-loop system.

B. Proposed Method

1. Step 1: Fast Discretization of CT Control System

Suppose that the control system to be redesigned is as shown in Fig. 1a. The CT control system provides satisfactory closed-loop performance. The CT plant $\bar{G}(s)$ may comprise actuator and sensor dynamics. Precede $\bar{G}(s)$ by a hold device H and place the ideal sampler S at the output of $\bar{G}(s)$. H and S are synchronized at the high rate of $1/T$. The transfer function of $S\bar{G}H$ is given by $G(z, T)$. Proceed similarly for controller $\bar{C}(s)$. Then there results a fast, single-rate

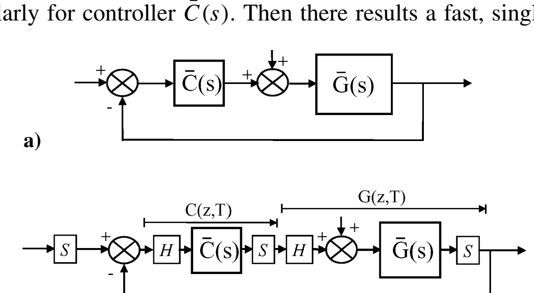

Fig. 1 Control system: a) CT and b) DT obtained by fast discretization.

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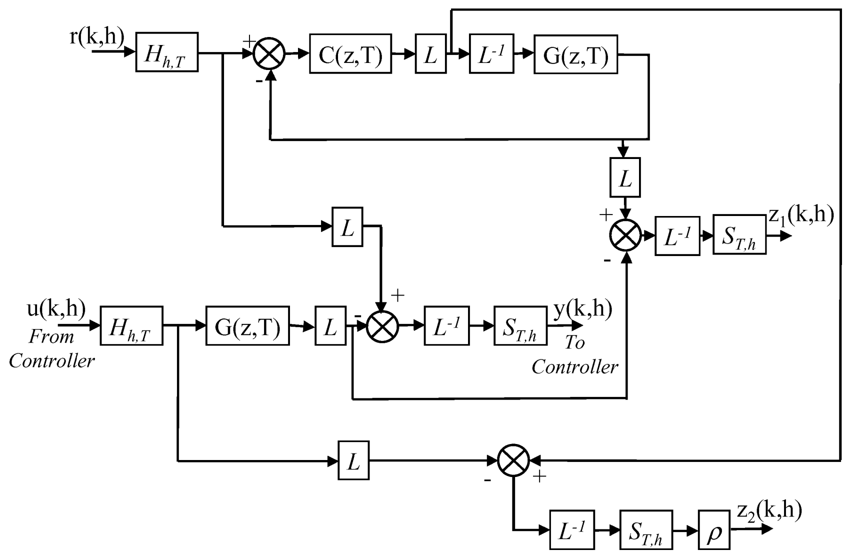

Fig. 2 Dual-rate generalized plant.

DT closed-loop system, shown in Fig. 1b. The selection of T depends on the design specifications and on the dynamics of the system under control.

2. Step 2: Dual-Rate Generalized Plant Modeling for Control Synthesis

If the fast DT controller $C(z,T)$ and DT plant $G(z,T)$ are known, a dual-rate generalized plant model can be constructed, as shown in Fig. 2. The generalized plant model in Fig. 2 is connected to a controller to be synthesized, $\hat{C}(z,h)$, which has output $u(k,h)$ and input $y(k,h)$. $\hat{C}(z,h)$ can be such that the induced norm from reference input $r(k,h)$ to outputs $z_1(k,h)$ and $z_2(k,h)$ is minimized. The dual-rate DT control system then provides an optimal approximation of the fast DT control system. It should be noted that the block diagram of Fig. 2 is intended for step-tracking purposes. In the figure, the gain $\rho \in \mathcal{R}$ is a weight on the difference in plant input signals that can be tuned by the designer.[5] In Fig. 2, L represents the DT lifting operation and L^{-1} is the inverse DT lifting.[9] DT lifting allows reformulating a multirate DT system as a single-rate DT system, although at the cost of increased input and output spaces.[9] For example, assuming $N=3$, lifting the DT scalar signal $x(k,T)$, given as

$$x(k,T) = \{x(0,T), x(1,T), x(2,T), x(3,T), \ldots\} \tag{1}$$

results in the DT vector signal $x^L(l,h)$, given by

$$x^L(l,h) = \left\{ \begin{bmatrix} x(0,T) \\ x(1,T) \\ x(2,T) \end{bmatrix}, \begin{bmatrix} x(3,T) \\ x(4,T) \\ x(5,T) \end{bmatrix}, \ldots \right\} \tag{2}$$

Note that $L^{-1}L = I$, so no information is lost when lifting and inverse lifting are successively applied on a DT signal.[5] The generalized plant of Fig. 2 is dual-rate due to the presence of $H_{h,T}$ and $S_{T,h}$. System $H_{h,T}$, defined as a dual-rate generalized hold[10] in this note, receives a DT input signal with period h and outputs a DT signal with period T. The simplest dual-rate generalized hold is the DT ZOH, which acts as an N-time repeater of its input signal. Systems $S_{T,h}$ are DT systems that take a DT input signal having period T and output a DT signal having period h. One such $S_{T,h}$ is the decimator that outputs every Nth input sample. Another class of dual-rate samplers $S_{T,h}$ are named averagers; that is, $S_{T,h}L^{-1}$ have an input–output relationship given by

$$w(k,h) = \begin{bmatrix} \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix} \underbrace{\begin{bmatrix} v(Nk,T) \\ \vdots \\ v(Nk+N-1,T) \end{bmatrix}}_{=v^L} \tag{3}$$

where $w(k,h)$ is the average value of the entries of the lifted input v^L at each time step $k \cdot h$. In this Note, averagers are placed at the $z_1(k,h)$ and $z_2(k,h)$ branches of the generalized plant, whereas the decimator is located at $y(k,h)$.

3. Step 3: Solution to DT H_2 Optimal Control Problem

The dual-rate generalized plant obtained in Step 2 can be placed in a closed loop with $\hat{C}(z,h)$, which is the DT controller to be designed. The dual-rate digital-redesign problem can then be formulated as follows: Given a fast DT control system, design a dual-rate DT control system such that its closed-loop step responses optimally match those of the fast DT control system in the sense that the cost function J, given by (4), is minimized:

$$J = \sum_{k=0}^{\infty} |z_1(k,h)|^2 + |z_2(k,h)|^2 \tag{4}$$

The minimization of J is a DT H_2 problem. With a unit DT impulse input $r(k,h)$ and an appropriate filter placed at the exogenous input channel to convert the impulse signal to a step, the DT H_2 problem consists in obtaining a controller $\hat{C}(z,h)$ such that the DT H_2 norm of the closed-loop system relating $r(k,h)$ to $[z_1(k,h), z_2(k,h)]^T$ is minimized or, equivalently, such that J is minimized.

Remarks: 1) It should be noted that the blocks $H_{h,T}$ and $S_{T,h}$ impact on the performance of the closed-loop system and that the selection of such blocks is an integral part of the design process. 2) The DT H_2 problem can be solved using algebraic Riccati equations or linear matrix inequalities (LMI) techniques, although the solution method constrains the selection of $H_{h,T}$ and $S_{T,h}$. 3) Controller $\hat{C}(z,h)$ is strictly proper.

III. Missile Autopilot

Consider a symmetrical, tail-controlled missile. Pitch-plane dynamics can be linearized at a given altitude and Mach number to yield the following linear models[11]:

$$\frac{q(s)}{\delta_p(s)} = \frac{M_\delta s + (M_\delta/V_M)Z_\alpha - (M_\alpha/V_M)Z_\delta}{s^2 + (Z_\alpha/V_M)s - M_\alpha}$$
$$\frac{a_z(s)}{\delta_p(s)} = \frac{Z_\delta s^2 + M_\delta Z_\alpha - M_\alpha Z_\delta}{s^2 + (Z_\alpha/V_M)s - M_\alpha} \tag{5}$$

where the input and the output of the transfer functions correspond to perturbation variables. In Eq. (5), a_z corresponds to missile acceleration along the z axis, q is the pitch rate resolved in the body coordinates, the fin deflection in pitch is δ_p, V_M is the magnitude of the missile velocity, M_δ, M_α, M_q are slopes of pitch moment per fin

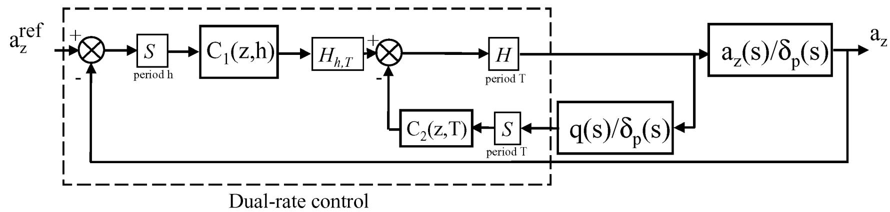

Fig. 3 Dual-rate missile autopilot.

deflection, angle of attack, and pitch rate, respectively, and Z_α, Z_δ are slopes of normal force per angle of attack and per fin deflection, respectively. Figure 3 presents the structure of the digital missile autopilot, where H and S are the ZOH and the ideal sampler, respectively, with period indicated below the block, and C_1 and C_2 are the outer- and inner-loop feedback controllers, respectively, which have to be designed. The inner-loop feedback of the digital missile autopilot is constrained to run at the sampling interval T, whereas the outer-loop feedback is fixed to period h.

A. Linearized Systems Simulations

Digital redesign is performed on a two-loop CT missile autopilot with controllers given as

$$\bar{C}_1(s) = 3.5 \times 10^{-6} + (1.28 \times 10^{-3})/s$$
$$\bar{C}_2(s) = -2.83 \times 10^{-2} \tag{6}$$

Tustin's method[1] is applied to $\bar{C}_1(s)$. The proposed dual-rate digital redesign is applied to the CT closed-loop system. In the process, ρ is set to unity. For the pair of sampling periods $T = 0.01$ and $h = 0.08$, the DT controllers $C_1(z, h)$ and $C_2(z, T)$ are

$$C_1(z,h) = \frac{0.55 \times 10^{-4} z + 0.48 \times 10^{-4}}{z-1}, \qquad C_2(z,T) = -2.83 \times 10^{-2} \qquad \text{(Tustin)}$$

$$\left.\begin{aligned} C_1(z,h) &= \frac{0.17 \times 10^{-3} z^5 - 0.024 \times 10^{-3} z^4 + 0.12 \times 10^{-3} z^3 - 0.0013 \times 10^{-3} z^2 + 1.36 \times 10^{-7} z - 7.2 \times 10^{-9}}{z^6 + 0.22 z^5 - 1.2 z^4 + 0.025 z^3 - 0.05 z^2 - 0.004 z + 0.0005} \\ C_2(z,T) &= -2.83 \times 10^{-2} \end{aligned}\right\} \quad \text{(Proposed)} \tag{7}$$

For the pair $T = 0.01$ and $h = 0.04$, the DT controllers are

$$C_1(z,h) = \frac{0.29 \times 10^{-4} z + 0.22 \times 10^{-4}}{z-1}, \qquad C_2(z,T) = -2.83 \times 10^{-2} \qquad \text{(Tustin)}$$

$$\left.\begin{aligned} C_1(z,h) &= \frac{0.099 \times 10^{-3} z^5 - 0.113 \times 10^{-3} z^4 + 0.066 \times 10^{-3} z^3 - 0.022 \times 10^{-3} z^2 + 0.0042 \times 10^{-3} z - 0.00035 \times 10^{-9}}{z^6 - 1.17 z^5 + 0.07 z^4 + 0.175 z^3 - 0.071 z^2 - 0.00187 z + 0.0021} \\ C_2(z,T) &= -2.83 \times 10^{-2} \end{aligned}\right\} \quad \text{(Proposed)}$$

$$\left.\begin{aligned} C_1(z,h) &= \frac{-0.11 \times 10^{-4} z^2 + 0.954 \times 10^{-4} z - 0.399 \times 10^{-4}}{z^2 - 1.002 z + 0.00209} \\ C_2(z,T) &= -2.83 \times 10^{-2} \end{aligned}\right\} \quad \text{(Proposed, reduced-order)} \tag{8}$$

where the proposed reduced-order controller is the result of a balanced-truncation[12] model reduction of the proposed sixth-order controller. As seen in (7) and (8), as N is made larger, $C_1(z, h)$ obtained with the proposed approach may become unstable, although the closed-loop system remains stable. Figures 4–6 show the responses of the DT and CT missile autopilots to a unit-step acceleration demand and a unit-step disturbance input for an altitude of 500 m and a speed of Mach 4. Three sampling-period pairs are studied. For brevity, simulation results obtained at a single operating point are shown. However, similar results have been obtained for a set of operating points. The DT control system obtained with the proposed technique provides the best step-tracking and disturbance-rejection performances for the three sampling pairs tested, as compared with the digital autopilot based on Tustin's method. The proposed reduced-order digital control law results in performances superior to those for Tustin's method for $N = 4$ and $N = 2$. Even though the overshoot has increased with a larger outer-loop sampling period h, the proposed method results in a stable closed-loop system that tracks the acceleration reference input and has an acceptable rise time. It should be pointed out that the step response obtained with the proposed method shows a one-sample delay, because the controller is strictly proper. As expected, a reduction in h improves the closed-loop tracking and disturbance-rejection performances. In fact, for a relatively small T, as $N \to 1$, the performances of the proposed and classical DT control systems approach that of the CT control system. It should be noted that Tustin's method yields an unstable closed-loop system for $h \geq 20T$ (with fixed $T = 0.01$), whereas the proposed digital redesign method preserves closed-loop stability even for such large sampling periods. However, reducing the order of the controller with the balanced-truncation model reduction method results in an unstable closed-loop system for $N = 8$.

B. Nonlinear 6DOF Airframe Dynamics Simulations

Acceleration command tracking of the nonlinear 6DOF missile dynamics described in Ref. 11 is illustrated with the gain-scheduled digital control. The proposed missile autopilot is applied along with gain scheduling of the controller-transfer-function coefficients. The scheduling variables are missile Mach number and altitude. To facilitate the implementation, at each operating point, controller order reduction is performed using balanced truncation[12] such that the DT controllers have order smaller than or equal to 3. Figure 7 shows the square-wave responses of the gain-scheduled dual-rate

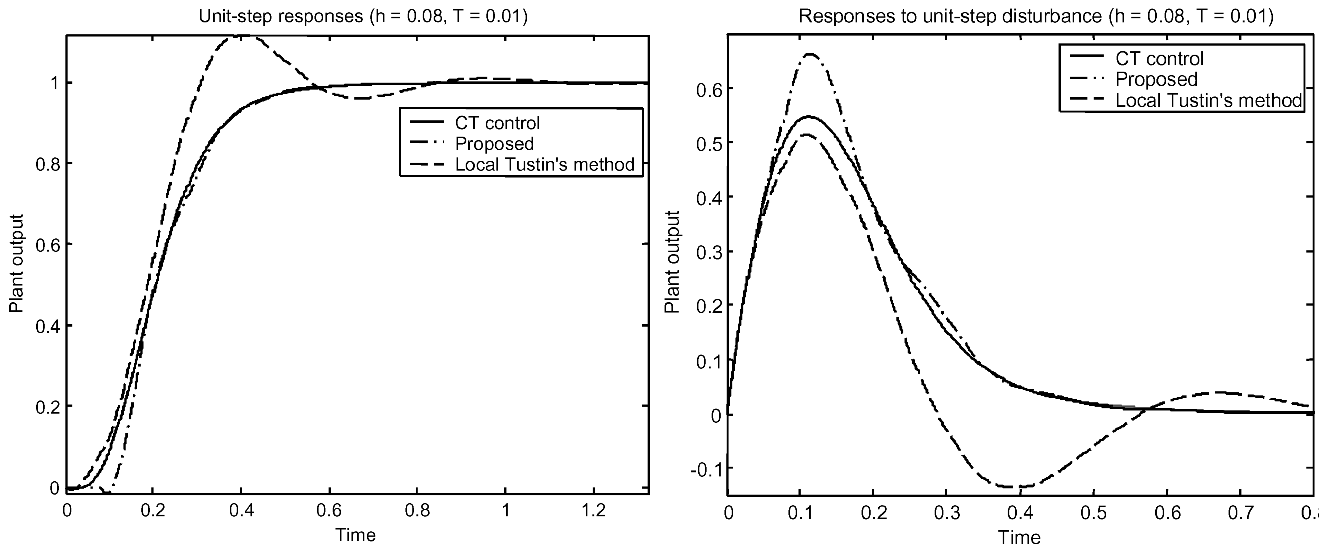

Fig. 4 Tracking and disturbance rejection for $T = 0.01$ and $N = 8$.

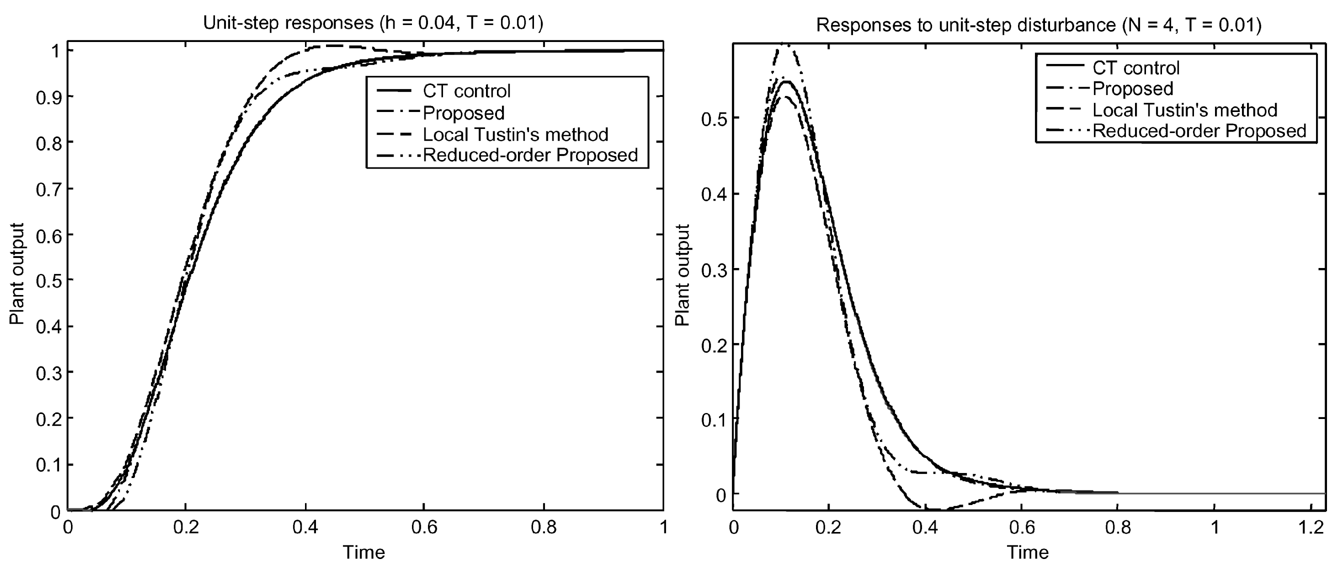

Fig. 5 Tracking and disturbance rejection for $T = 0.01$ and $N = 4$.

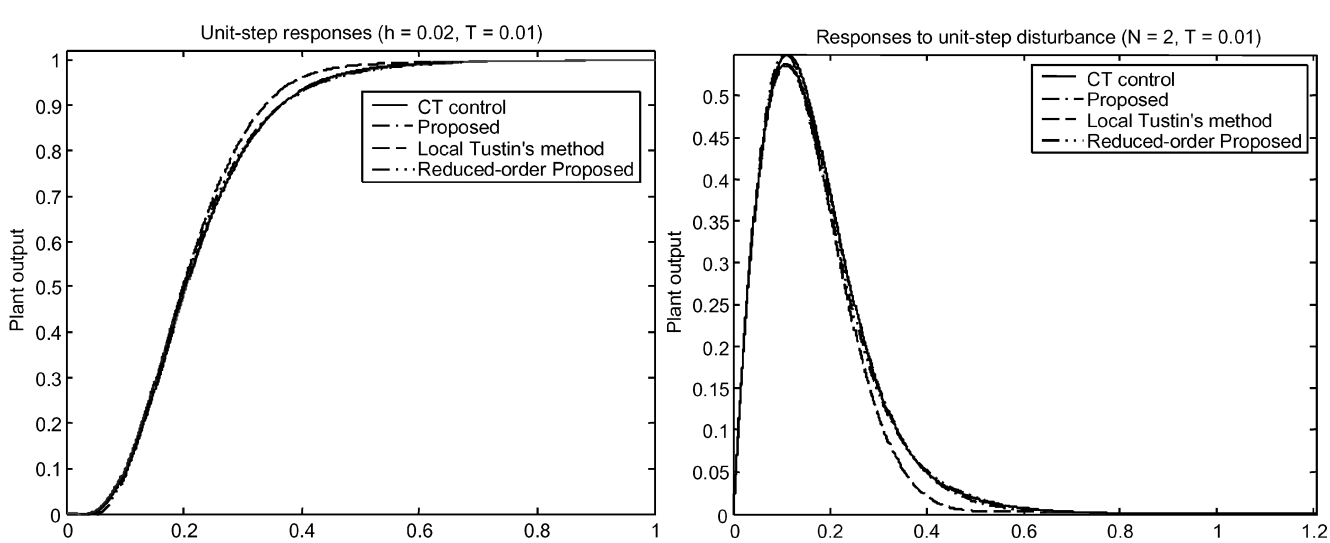

Fig. 6 Tracking and disturbance rejection for $T = 0.01$ and $N = 2$.

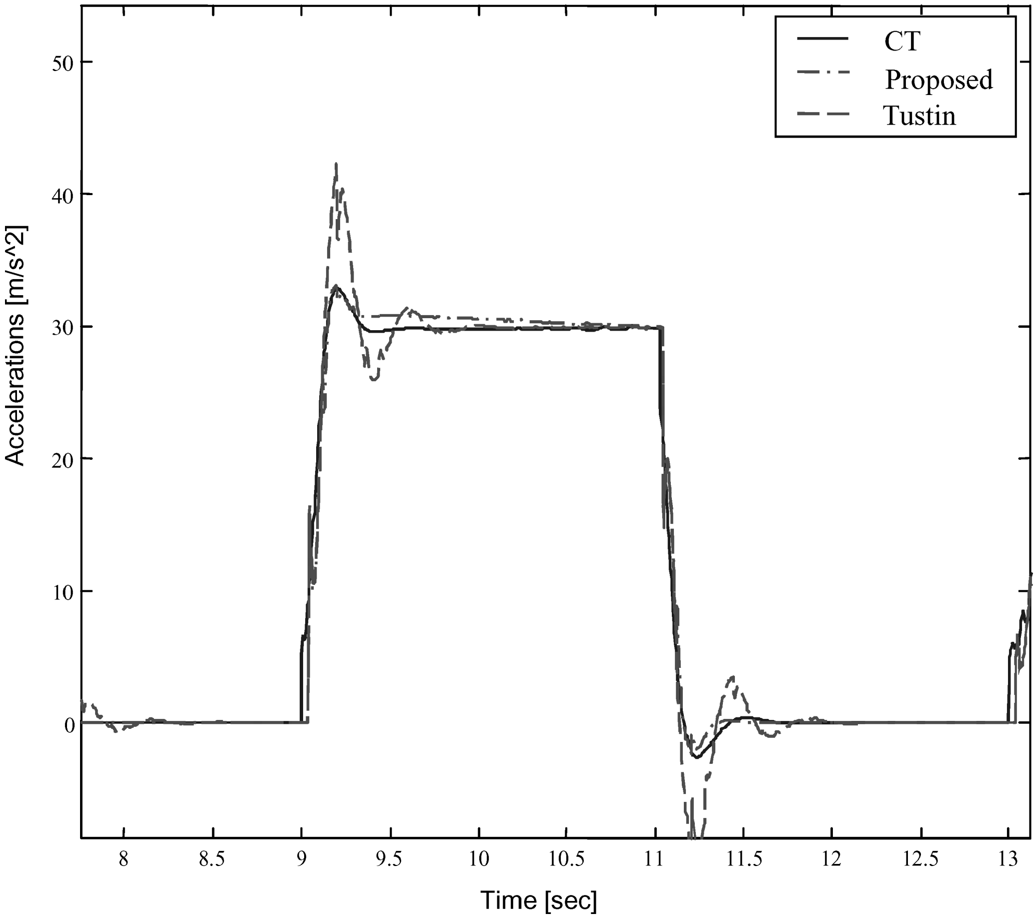

Fig. 7 Acceleration tracking obtained with the gain-scheduled autopilots

missile autopilots for $T = 0.01$ s ($N = 15$) and that obtained with the gain-scheduled CT missile autopilot over a prescribed time window. As in the linear case, the proposed gain-scheduled dual-rate digital missile autopilot has a satisfactory closed-loop performance, as opposed to the digital autopilot obtained with Tustin's method, which results in excessive overshoot.

IV. Conclusions

This Note proposes an optimal dual-rate digital-redesign method that relies on the solution of a dual-rate H_2 discrete-time control problem. The proposed technique results in either single- or dual-rate digital control systems, depending on the designer's needs, with satisfactory closed-loop performance in terms of reference input tracking and disturbance rejection, over an extended range of sampling rates compared with other widely used methods of digital redesign. In the future, the digital-redesign method could be extended to include consideration of stability and performance robustness.

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